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Multidimensional generalized coherent states

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Abstract

Generalized coherent states were presented recently for systems with one degree of freedom having discrete and/or continuous spectra. We extend that definition to systems with several degrees of freedom, give some examples and apply the formalism to the model of two-dimensional fermion gas in a constant magnetic field.

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1. Introduction

One recent generalization of coherent states [1, 2] is based on a set of three requirements introduced by John Klauder some years ago [3], namely normalization, continuity in the parameter(s) and a resolution of the unity. For the sake of completeness, we outline this formalism here. Based on the required properties, one can say that, given a finite or separable infinite-dimensional Hilbert space with orthonormal basis denoted as $\{|n\rangle\}_{n\in\mathbb{N}}$, a superposition of the type

$$|J,\gamma\rangle = \frac{1}{\sqrt{\mathcal{N}(J)}}\sum_{n=0}^{\varpi}\frac{J^{n/2}}{\sqrt{\rho_n}}\,\mathrm{e}^{-\mathrm{i}\gamma e_n}|n\rangle \qquad 0\leqslant J<R \quad \gamma\in\mathbb{R} \quad \varpi\leqslant\infty \tag{1}$$

is a coherent state if

$$\mathcal{N}(J) = \sum_{n=0}^{\varpi}\frac{J^n}{\rho_n} \tag{2}$$

is convergent for $J < R$ and if the moment problem

$$\int_0^R J^n\frac{W(J)}{\mathcal{N}(J)}\,\mathrm{d}J = \rho_n \tag{3}$$

admits a positive solution for $W(J)$ (R can of course be infinite). Many different moment problems of this type are presented and solved for instance in [4]. Knowledge of $W(J)$ allows a resolution of identity in terms of the coherent states:

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \mathrm{d}\gamma \int_0^R \mathrm{d}J\, W(J)|J,\gamma\rangle\langle J,\gamma| = \mathbb{I}. \tag{4}$$

We now suppose that the kets $|n\rangle$ are eigenvectors of a self-adjoint operator H with eigenvalues ωe_n,

$$H|n\rangle = \omega e_n|n\rangle \tag{5}$$

ω being some positive constant and $\{e_n\}$ being a strictly increasing sequence of numbers with $e_0 = 0$. We thus obtain a property that we call evolution stability (temporal evolution if H has a Hamiltonian meaning):

$$\mathrm{e}^{-\mathrm{i}Ht}|J,\gamma\rangle = |J,\gamma+\omega t\rangle. \tag{6}$$

In previous works the relation $\langle J,\gamma|H|J,\gamma\rangle = \omega J$ (or some variant of it, see [2]), called the action identity, was imposed. In order to obtain it from our definition (1) we have to set

$$\rho_n = e_1 e_2, \ldots, e_n \qquad \text{for} \quad n \geqslant 1 \qquad \rho_0 = 1. \tag{7}$$

Different choices for the function ρ_n are possible and will give different mean values for H and also different moment problems. The variable J was called the 'action variable' but we hereafter call it the coherence variable.

In what follows we shall present a few explicit examples, based on the simplest and most popular coherent states.

1.1. Examples

One should note that all the following cases have in common the linear nature of the spectrum, which allows a special grouping of the J and γ variables leading to an analytical formulation of the Fock–Bargmann type.

1.1.1. Harmonic oscillator. The space of states of the harmonic oscillator is an infinite-dimensional Hilbert space in which its stationary Schrödinger equation reads ($\hbar = 1$)

$$H|n\rangle = \omega n|n\rangle \tag{8}$$

(we consider a shifted Hamiltonian to lower the zero-point energy to zero). Therefore, equation (1) becomes

$$|J,\gamma\rangle = \mathrm{e}^{-J/2} \sum_{n=0}^{\infty} \frac{J^{n/2}}{\sqrt{n!}} \mathrm{e}^{-\mathrm{i}n\gamma}|n\rangle. \tag{9}$$

Identifying $\sqrt{J}\,\mathrm{e}^{-\mathrm{i}\gamma} \equiv z$ we have the canonical coherent states, also called the Glauber–Klauder–Sudarshan (GKS) states.

These states are overcomplete with weight function $W(J) = \frac{1}{2\pi}$:

$$\frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\gamma \int_0^{\infty} \mathrm{d}J |J,\gamma\rangle\langle J,\gamma| = \sum_{n=0}^{\infty} |n\rangle\langle n| = \mathbb{I}. \tag{10}$$

1.1.2. The case of $su(1,1)$. Another interesting choice for ρ_n is based on the binomial coefficient:

$$\rho_n = \binom{\nu+n}{n}^{-1} \qquad 1 \leqslant \nu \in \mathbb{N}. \tag{11}$$

In this case we have

$$|\nu; J, \gamma\rangle = (1-J)^{(\nu+1)/2} \sum_{n=0}^{\infty} \sqrt{\binom{\nu+n}{n}} J^{n/2} \, \mathrm{e}^{-\mathrm{i}n\gamma} |n\rangle \tag{12}$$

and the normalization condition imposes $0 \leqslant J < 1$. These states can be identified with the Perelomov coherent states for the $su(1,1)$ algebra in its discrete series representation U^ν where $\nu \in \mathbb{N}^*$ [8]. Here we recall that the three generators of this algebra obey the commutation relations

$$[K_0, K_\pm] = \pm K_\pm \qquad [K_+, K_-] = -2K_0 \tag{13}$$

and in the involved discrete series representation we have

$$K_0|\nu, n\rangle = \left(\frac{\nu+1}{2} + n\right)|\nu, n\rangle \qquad \nu \geqslant 1 \quad n \geqslant 0. \tag{14}$$

(This notation is slightly different from the standard one.) Similar to the previous case, the grouping of the two parameters into the complex number $z = \sqrt{J}\,\mathrm{e}^{-\mathrm{i}\gamma}$ leads to a Fock–Bargmann formalism. However, the important difference with the oscillator (or Weyl–Heisenberg) case lies in the fact that z is now restricted to the open unit disc.

The weight function for these states is $W(J) = \frac{\nu}{2\pi(1-J)^2}$ and the overcompleteness relation holds in the unit circle:

$$\frac{\nu}{2\pi} \int_0^{2\pi} \mathrm{d}\gamma \int_0^1 \frac{\mathrm{d}J}{(1-J)^2} |\nu; J, \gamma\rangle\langle\nu; J, \gamma| = \sum_{n=0}^{\infty} |n\rangle\langle n| = \mathbb{I}. \tag{15}$$

1.1.3. The case of $su(2)$. We can also apply this formalism to a finite-dimensional system, for example, a unitary irreducible representation of $su(2)$, a case in which we also have a complex Fock–Bargmann structure and for which we have the equation $L_z|j,m\rangle = m|j,m\rangle$, $j \in \mathbb{N}/2, -j \leqslant m \leqslant j$. The condition $e_n \geqslant 0$ demands that we introduce the shifted operator $\widetilde{L}_z = L_z + j$ and the states $|n\rangle \equiv |j,m\rangle$ such that $n = j+m$. Therefore, $\widetilde{L}_z|n\rangle = n|n\rangle$ and we can write

$$|z\rangle = \frac{1}{\sqrt{\mathcal{N}(|z|^2)}} \sum_{n=0}^{2j} \frac{z^n}{\sqrt{\rho_n}} |n\rangle \qquad z = \sqrt{J}\,\mathrm{e}^{-\mathrm{i}\gamma}. \tag{16}$$

If we choose

$$\rho_n = \binom{2j}{n}^{-1} \tag{17}$$

we have

$$\mathcal{N}(|z|^2) = \sum_{n=0}^{2j} \binom{2j}{n} |z|^{2n} = (1+|z|^2)^{2j} \tag{18}$$

and therefore

$$|z\rangle = \frac{1}{(1+|z|^2)^j} \sum_{m=-j}^{j} \sqrt{\binom{2j}{j+m}} z^{j+m} |j, m\rangle \tag{19}$$

which are the usual $su(2)$ coherent states. We can use points of the unit sphere rather than those on the plane to label these states, through the projection $z = \tan(\theta/2)\,e^{-i\varphi} = \sqrt{J}\,e^{-i\gamma}$ ($\varphi \equiv \gamma$ is the azimuth and θ is the latitude). The states are then denoted by $|j;\theta,\varphi\rangle$ and are known as Bloch states in the literature. They are of course overcomplete, as we can see from the formula

$$\frac{2j+1}{4\pi}\int |j;\theta,\varphi\rangle\langle j;\theta,\varphi|\,dS = \frac{2j+1}{\pi}\int \frac{d^2z}{(1+|z|^2)^2}|z\rangle\langle z| = \sum_{m=-j}^{j}|j,m\rangle\langle j,m| = \mathbb{I}_{2j+1} \tag{20}$$

where $dS = \sin\theta\,d\theta\,d\varphi$ and $\mathbb{I}_q$ is the identity in the q-dimensional Hermitian space $\mathbb{C}^q$ carrying the involved UIR of $su(2)$.

2. Symbols and Berezin–Lieb inequalities

Berezin [6] and Lieb [7] separately introduced the concepts of upper and lower symbols of an operator A, respectively $\hat{A}$ and $\check{A}$. They are defined through the relations

$$A = \int d\mu(z)\hat{A}(z)|z\rangle\langle z| \qquad \check{A}(z) = \langle z|A|z\rangle$$

where $|z\rangle$ is a set of complete (or overcomplete) normalized quantum states that (usually, but not necessarily) provide a resolution of the unity. Note that given an operator A its upper symbol is not unique in general. Let us give some of these symbols for the three examples we just presented.

- For the harmonic oscillator and in terms of the corresponding creation and annihilation operators,

$$\langle z|a^\dagger a|z\rangle = |z|^2 \tag{21}$$

$$a^\dagger a = \int d^2z(|z|^2 - 1)|z\rangle\langle z|. \tag{22}$$

- For the algebra $su(2)$,

$$\langle j;\theta,\varphi|L_z|j;\theta,\varphi\rangle = j\cos\theta \tag{23}$$

$$L_z = \frac{2j+1}{4\pi}\int dS(j+1)\cos\theta|l;\theta,\varphi\rangle\langle j;\theta,\varphi|. \tag{24}$$

- For the algebra $su(1,1)$,

$$\langle\nu; J,\gamma|K_0|\nu; J,\gamma\rangle = \frac{\nu+1}{2}\left(\frac{1+J}{1-J}\right) \tag{25}$$

$$K_0 = \frac{1}{2\pi}\int_0^{2\pi} d\gamma \int_0^1 \frac{\nu\,dJ}{(1-J)^2}\left(\frac{\nu-1}{2}\right)\left(\frac{1+J}{1-J}\right)|\nu; J,\gamma\rangle\langle\nu; J,\gamma|. \tag{26}$$

In the case $\nu = 1$ the last equation does not apply and has to be replaced by

$$K_0 = \lim_{\epsilon\to 0^+}\frac{1}{2\pi}\int_0^{2\pi} d\gamma\int_0^1 dJ\frac{\delta(1-\epsilon-J)}{(1-J)^2}|\nu=1; J,\gamma\rangle\langle\nu=1; J,\gamma|. \tag{27}$$

It can be proved (see [7]) that, given any convex function g, the following inequalities (called the Berezin–Lieb inequalities) hold:

$$\int \mathrm{d}\mu(z) g(\check{A}) \leqslant \mathrm{Tr}(g(A)) \leqslant \int \mathrm{d}\mu(z) g(\hat{A}). \tag{28}$$

As an application of our formalism, we shall present in section 4 an evaluation of the Berezin–Lieb (BL) inequalities for the thermodynamic potential of a two-dimensional electron gas in a constant perpendicular magnetic field.

At this point, an interesting question can be addressed: given a separable Hilbert space with an orthonormal basis $\{|n\rangle, n \in \mathbb{N}\}$, the related number operator N such that $N|n\rangle = n|n\rangle$, one can build the families of states (9) and (12). How are the associated BL inequalities with $A = N$ related? The answer is a simple coordinate change. When dealing with the family (12) one should interpret N as $K_0 - (\nu+1)/2$ and the BL inequalities are

$$\int_0^1 \frac{\nu\,\mathrm{d}J}{(1-J)^2} g\left(\frac{(\nu+1)J}{1-J}\right) \leqslant \mathrm{Tr}(g(N)) \leqslant \int_0^1 \frac{\nu\,\mathrm{d}J}{(1-J)^2} g\left(\frac{(\nu-1)J}{1-J}\right) \tag{29}$$

where g is any convex function. The transformations

$$x_\pm = \frac{(\nu \pm 1)J}{1-J} \tag{30}$$

take (29) to

$$\frac{\nu}{\nu+1} \int_0^\infty g(x_+)\,\mathrm{d}x_+ \leqslant \mathrm{Tr}(g(N)) \leqslant \frac{\nu}{\nu-1} \int_0^\infty g(x_-)\,\mathrm{d}x_-. \tag{31}$$

It is evident that if we had used family (9) we would have reached the same result, except for the ν-dependence. This dependence shows that the canonical coherent states are more suited for calculating BL inequalities than any of the families (12). A similar observation can be made for the $su(2)$ coherent states and one can see that in the limit $j \to \infty$ they give the same results as the canonical ones.

3. Generalization

We now want to generalize expression (1) to the case of several degrees of freedom, that is, when the basis states are written as $|n_1, n_2, \ldots, n_r\rangle \equiv |\mathbf{n}\rangle$, $r \geqslant 2$.

Let us begin by an example extending the above $su(2)$ construction. Consider the Hilbert space $\mathcal{H} = \oplus_{j\in\mathbb{N}/2}\mathcal{H}_j$, orthogonal direct sum of all Hermitian spaces $\mathcal{H}_j = \mathbb{C}^{2j+1}$ carrying unitary irreducible representations of $su(2)$. For each representation U^j we have the family $\{|j;\theta,\varphi\rangle\}$ satisfying (20), which in this case we call 'resolution of the orthogonal projector' $\mathbb{I}_{2j+1}$. To get an overcomplete family of states solving the unity on the large Hilbert space $\mathcal{H}$ one could, for example, define

$$|J_1, J_2, \gamma_1, \gamma_2\rangle = \mathrm{e}^{-J_1/2} \sum_{2j\in\mathbb{N}} \frac{J_1^j\,\mathrm{e}^{-\mathrm{i}2j\gamma_1}}{\sqrt{(2j)!}} |j;\theta,\varphi\rangle \equiv |J_1, \gamma_1, \theta, \varphi\rangle \tag{32}$$

where $\tan(\theta/2)\,\mathrm{e}^{-\mathrm{i}\varphi} = \sqrt{J_2}\,\mathrm{e}^{-\mathrm{i}\gamma_2}$. One should note the 'GKS-like' character of this superposition. These states obey

$$\int_0^{2\pi} \mathrm{d}\gamma_1 \int_0^\infty \mathrm{d}J_1 \int \mathrm{d}S\, W(J_1) |J_1, \gamma_1, \theta, \varphi\rangle\langle J_1, \gamma_1, \theta, \varphi| = \mathbb{I} \tag{33}$$

with $W(J_1) = \frac{J_1}{8\pi^2}$. Note that it is interesting in itself to divide the large Hilbert space into its bosonic and fermionic parts,

$$\mathcal{H} = \mathcal{H}_{\mathrm{bos}} \oplus \mathcal{H}_{\mathrm{ferm}} \tag{34}$$

and to give the explicit form of the weight functions $W_{\rm bos}$ and $W_{\rm ferm}$ involved in the resolution of the respective projectors $\mathbb{I}_{\rm bos}$ and $\mathbb{I}_{\rm ferm}$. The result is

$$|J_1, \gamma_1, \theta, \varphi\rangle_{\text{-bos}} = \mathrm{e}^{-J_1/2} \sum_j \frac{J_1^{j/2}\,\mathrm{e}^{-\mathrm{i}j\gamma_1}}{\sqrt{j!}} |j;\theta,\varphi\rangle \qquad j = 0, 1, 2, \ldots \tag{35}$$

$$|J_1, \gamma_1, \theta, \varphi\rangle_{\text{-ferm}} = \mathrm{e}^{-J_1/2} \sum_j \frac{J_1^{(2j-1)/4}\,\mathrm{e}^{-\mathrm{i}(2j-1)\gamma_1/2}}{\sqrt{\left(\frac{2j-1}{2}\right)!}} |j;\theta,\varphi\rangle \qquad j = \frac{1}{2}, \frac{3}{2}, \ldots \tag{36}$$

$$W_{\rm bos}(J_1) = \frac{2J_1 - 1}{8\pi^2} \tag{37}$$

$$W_{\rm ferm}(J_1) = \frac{2J_1}{8\pi^2}. \tag{38}$$

Based on this simple example, we obtain multidimensional generalizations of the standard one-dimensional coherent states, possibly in a recursive fashion. First of all, we assume we have a complete set of r commuting observables satisfying the eigenvalue equations:

$$A_i|\mathbf{n}\rangle = \omega_i e_i(\mathbf{n})|\mathbf{n}\rangle. \tag{39}$$

So, we could deal with the following general form of corresponding coherent states:

$$|\mathbf{J}, \boldsymbol{\gamma}\rangle = \frac{1}{\sqrt{\mathcal{N}(\mathbf{J})}} \sum_{\{n\}} \frac{\mathbf{J}^{\mathbf{n}/2}}{\sqrt{\rho(\mathbf{n})}} \mathrm{e}^{-\mathrm{i}\boldsymbol{\gamma}\cdot\mathbf{e}(\mathbf{n})}|\mathbf{n}\rangle \tag{40}$$

where the sum runs over all possible values of the variables n_i, $\mathcal{N}$ is a normalization factor and $\rho(\mathbf{n})$ is an arbitrary positive function of all the indices. The expressions $\mathbf{J}^{\mathbf{n}/2}$ and $\boldsymbol{\gamma}\cdot\mathbf{e}(\mathbf{n})$ stand for $\prod_{i=1}^r J_i^{n_i/2}$ and $\gamma_1 e_1(\mathbf{n}) + \cdots + \gamma_r e_r(\mathbf{n})$, respectively. We could add specific conditions to definition (40) as was done in [1, 2], but we will rather adopt a more intuitive approach, based on recursivity.

We can first introduce coherence variables for the rth degree of freedom:

$$|n_1, n_2, \ldots, J_r, \gamma_r\rangle = \frac{1}{\sqrt{\mathcal{N}_r(J_r)}} \sum_{n_r} \frac{J_r^{n_r/2}}{\sqrt{\rho_r}} \mathrm{e}^{-\mathrm{i}\gamma_r e_r(\mathbf{n})}|\mathbf{n}\rangle \tag{41}$$

where the sum runs over all possible values of n_r and both the norm $\mathcal{N}_r(J_r)$ and the function ρ_r may depend on the remaining indices. These states should satisfy a resolution of the orthogonal projector on the subspace defined by fixing $n_1, n_2, \ldots, n_{r-1}$ (we shall impose the adapted Klauder conditions at each step):

$$\int \mathrm{d}\mu(J_r, \gamma_r)|n_1, n_2, \ldots, J_r, \gamma_r\rangle\langle n_1, n_2, \ldots, J_r, \gamma_r| = \sum_{n_r} |\mathbf{n}\rangle\langle\mathbf{n}| = \mathbb{I}_{n_1, n_2, \ldots, n_{r-1}}. \tag{42}$$

Now, if we suppose that the dependence of the ρ_i and e_i on the r-uple $\boldsymbol{n}$ is organized in an hierarchical fashion as $\rho_i(\boldsymbol{n}) = \rho_i(n_1, n_2, \ldots, n_i)$ and $e_i(\boldsymbol{n}) = e_i(n_1, n_2, \ldots, n_i)$, we can proceed to associate a coherence variable with each degree of freedom until we get

$$|\boldsymbol{J}, \boldsymbol{\gamma}\rangle = \frac{1}{\sqrt{\mathcal{N}_1}} \sum_{n_1} \frac{J_1^{n_1/2}}{\sqrt{\rho_1}} \mathrm{e}^{-\mathrm{i}\gamma_1 e_1} \frac{1}{\sqrt{\mathcal{N}_2}} \sum_{n_2} \frac{J_2^{n_2/2}}{\sqrt{\rho_2}} \mathrm{e}^{-\mathrm{i}\gamma_2 e_2} \cdots \frac{1}{\sqrt{\mathcal{N}_r}} \sum_{n_r} \frac{J_r^{n_r/2}}{\sqrt{\rho_r}} \mathrm{e}^{-\mathrm{i}\gamma_r e_r}|\mathbf{n}\rangle \tag{43}$$

where $\mathcal{N}_i$ stands for $\mathcal{N}_i(J_i, J_{i+1}, \ldots, J_r; n_1, n_2, \ldots, n_{i-1})$.

Equation (39) guarantees stability under the action of the group generated by all operators A_i. The choice of the functions ρ_i will determine their expectation values. One should keep in mind the possible dependence of ρ_i and e_i on the indices n_j, $j < i$ and it is evident that if such dependence is not present then one will end up with simple tensor products of states of type (1).

3.1. Examples

In order to illustrate the formalism, let us deal with a simple case in which we have two degrees of freedom, $r = 2$. In this case equation (43) reduces to

$$|J_1, J_2, \gamma_1, \gamma_2\rangle = \frac{1}{\sqrt{\mathcal{N}_1(J_1, J_2)}} \sum_{n_1} \frac{J_1^{n_1/2}}{\sqrt{\rho_1(n_1)}} \mathrm{e}^{-\mathrm{i}\gamma_1 e_1} \frac{1}{\sqrt{\mathcal{N}_2(n_1, J_2)}} \times \sum_{n_2} \frac{J_2^{n_2/2}}{\sqrt{\rho_2(n_1, n_2)}} \mathrm{e}^{-\mathrm{i}\gamma_1 \, e_2} |n_1, n_2\rangle. \tag{44}$$

We have already defined one kind of generalized coherent state for such a space in (32). We now present two others.

3.1.1. GKS–GKS. The standard choice $\rho_{n_i} = n_i!$, $e_i = n_i$, $i = 1, 2$, yields the tensor product of two independent GKS coherent states:

$$|z_1, z_2\rangle = \mathrm{e}^{-(|z_1|^2+|z_2|^2)/2} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \frac{z_1^{n_1}}{\sqrt{n_1!}} \frac{z_2^{n_2}}{\sqrt{n_2!}} |n_1, n_2\rangle \qquad z_j = \sqrt{J_j}\, \mathrm{e}^{-\mathrm{i}\gamma_j}. \tag{45}$$

These states are of course overcomplete and their weight function is simply $1/\pi^2$.

3.1.2. GKS-su$(1, 1)$. In this case we introduce coherence variables for the first degree of freedom in the following way:

$$|n_1, J_2, \gamma_2\rangle = (1 - J_2)^{(n_1+2)/2} \sum_{n_2=0}^{\infty} \sqrt{\binom{n_1 + n_2 + 1}{n_2}} J_2^{n_2/2} \mathrm{e}^{-\mathrm{i}\gamma_2 n_2} |n_1, n_2\rangle. \tag{46}$$

In this step we have, as in (15), a resolution of the projector $\mathbb{I}_{n_1} = \sum_{n_2=0}^{\infty} |n_1, n_2\rangle\langle n_1, n_2|$. We now introduce the second pair of coherence variables again in a 'GKS-like' manner:

$$|J_1, J_2, \gamma_1, \gamma_2\rangle = \mathrm{e}^{-J_1/2} \sum_{n_1=0}^{\infty} \frac{J_1^{n_1/2} \mathrm{e}^{-\mathrm{i}\gamma_1 n_1}}{\sqrt{n_1!}} |n_1, J_2, \gamma_2\rangle. \tag{47}$$

The complete resolution of identity now reads

$$\mathbb{I} = \int \mathrm{d}\mu(J_1, J_2, \gamma_1, \gamma_2) |J_1, J_2, \gamma_1, \gamma_2\rangle\langle J_1, J_2, \gamma_1, \gamma_2| \tag{48}$$

where

$$\int \mathrm{d}\mu(J_1, J_2, \gamma_1, \gamma_2) = \int_0^{2\pi} \mathrm{d}\gamma_1 \int_0^{2\pi} \mathrm{d}\gamma_2 \int_0^{\infty} \mathrm{d}J_1 \int_0^1 \mathrm{d}J_2 W(J_1, J_2) \tag{49}$$

$$W(J_1, J_2) = \frac{1}{4\pi^2} \frac{J_1}{(1 - J_2)^2}. \tag{50}$$

4. Application to 2D magnetism

The Hamiltonian for two-dimensional spinless electrons confined by an isotropic harmonic potential and submitted to a constant magnetic field **B** is written as

$$H = \frac{1}{2m} \left(\mathbf{P} + \frac{e}{c}\mathbf{A}\right)^2 + \frac{1}{2} m\omega_0^2 \mathbf{R}^2 \tag{51}$$

where Coulomb interactions are neglected. In the symmetric gauge $\mathbf{A} = \frac{1}{2}\mathbf{B} \times \mathbf{R}$ this Hamiltonian can be expressed as a sum of a two-dimensional isotropic harmonic oscillator and an angular momentum operator

$$H = \left(\frac{1}{2m}P_x^2 + \frac{1}{8}m\omega^2 X^2\right) + \left(\frac{1}{2m}P_y^2 + \frac{1}{8}m\omega^2 Y^2\right) + \frac{\omega_c}{2}L_0 \equiv H_0 + L_z \tag{52}$$

where $\omega_c = eB/mc$ is the cyclotron frequency, $\omega = \sqrt{\omega_c^2 + 4\omega_0^2}$ and $L_0 = XP_y - YP_x$.

Instead of directly using the oscillator annihilation operators,

$$a_x = \frac{1}{\sqrt{2}}\left(\frac{X}{l_0} + \frac{\mathrm{i}l_0}{\hbar}P_x\right) \qquad a_y = \frac{1}{\sqrt{2}}\left(\frac{Y}{l_0} + \frac{\mathrm{i}l_0}{\hbar}P_y\right) \tag{53}$$

one can work with two other ones, which are linear superpositions of a_x and a_y:

$$a_1 = \frac{1}{\sqrt{2}}(a_x - \mathrm{i}a_y) \qquad a_2 = \frac{1}{\sqrt{2}}(a_x + \mathrm{i}a_y) \tag{54}$$

where $l_0 = \sqrt{2\hbar/m\omega}$. Note that a_1 and a_2 are bosonic operators: $\left[a_1, a_1^\dagger\right] = \mathrm{I} = \left[a_2, a_2^\dagger\right]$. The operators H_0 and L_z can be simply expressed in terms of the number operators $N_1 = a_1^\dagger a_1$ and $N_2 = a_2^\dagger a_2$ as

$$H_0 = \frac{\hbar\omega}{2}(N_1 + N_2 + 1) \qquad L_z = \frac{\hbar\omega_c}{2}(N_1 - N_2). \tag{55}$$

The eigenvectors of the total Hamiltonian are tensor products of single Fock oscillator states:

$$|n_1, n_2\rangle = \frac{1}{\sqrt{n_1!n_2!}}\left(a_1^\dagger\right)^{n_1}\left(a_2^\dagger\right)^{n_2}|0, 0\rangle. \tag{56}$$

In the following we will use the three families of generalized coherent states presented above in order to obtain Berezin–Lieb inequalities for the thermodynamic potential associated with the Hamiltonian (52). This system has already been considered in [5], where the authors derived an exact analytic result for the thermodynamic potential. Our approach yields only bounds to this quantity, and recovers the results of [5] in some special limits.

4.1. Harmonic oscillator symmetry

We first deal with the tensor product states (45):

$$|z_1, z_2\rangle = \exp\left[-\frac{1}{2}(|z_1|^2 + |z_2|^2)\right]\sum_{n_1,n_2}\frac{z_1^{n_1}}{\sqrt{n_1!}}\frac{z_2^{n_2}}{\sqrt{n_2!}}|n_1, n_2\rangle. \tag{57}$$

In this case the upper and lower symbols for the total Hamiltonian read

$$2\hat{H} = \hbar\omega(|z_1|^2 + |z_2|^2 - 1) + \hbar\omega_c(|z_1|^2 - |z_2|^2) = 2\check{H} - 2\hbar\omega. \tag{58}$$

The Berezin–Lieb inequalities for the thermodynamic potential $\Omega = -\frac{1}{\beta}\,\mathrm{Tr}\ln(1 + \mathrm{e}^{-\beta(H-\mu)})$, where μ is the chemical potential and $\beta = 1/k_BT$, are given by

$$-\frac{1}{\beta\pi^2}\int \ln(1 + \mathrm{e}^{-\beta(\hat{H}-\mu)})\,\mathrm{d}^2z_1\,\mathrm{d}^2z_2 \leqslant \Omega \leqslant -\frac{1}{\beta\pi^2}\int \ln(1 + \mathrm{e}^{-\beta(\check{H}-\mu)})\,\mathrm{d}^2z_1\,\mathrm{d}^2z_2. \tag{59}$$

By making the substitutions $u = \hbar\beta[|z_1|^2(\omega + \omega_c) + |z_2|^2(\omega - \omega_c)]$, $v = \hbar\beta|z_1|^2(\omega + \omega_c)$, they reduce to

$$-\frac{1}{\beta}\int_0^\infty \mathrm{d}u\int_0^u \mathrm{d}v\,\ln(1 + \kappa_+\mathrm{e}^{-u}) \leqslant \Omega \leqslant -\frac{1}{\beta}\int_0^\infty \mathrm{d}u\int_0^u \mathrm{d}v\,\ln(1 + \kappa_-\mathrm{e}^{-u}) \tag{60}$$

and, after integrating by parts, eventually become

$$\phi(\kappa_+) \leqslant \Omega \leqslant \phi(\kappa_-) \tag{61}$$

where $\kappa_\pm = \mathrm{e}^{\beta(\mu \pm \hbar\omega/2)}$. The function ϕ is given by

$$\begin{aligned}\phi(\kappa) &= -\frac{\kappa}{2\beta(\beta\hbar\omega_0)^2}\int_0^\infty \frac{u^2\,\mathrm{e}^{-u}}{1+\kappa\,\mathrm{e}^{-u}}\,\mathrm{d}u \\ &= \frac{1}{\beta(\beta\hbar\omega_0)^2}\begin{cases} F_3(-\kappa) & \text{for} \quad \kappa \leqslant 1 \\ F_3(-\kappa^{-1}) - \frac{(\ln\kappa)^3}{6} - \frac{\pi^2\ln\kappa}{6} & \text{for} \quad \kappa > 1\end{cases}\end{aligned}$$

with

$$F_s(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^s}. \tag{62}$$

We can use inequalities (61) to study extreme regimes. For very high chemical potential with respect to the quantum $\hbar\omega/2$, or alternatively in a semiclassical regime, we have $\kappa_\pm \approx \mathrm{e}^{\beta\mu} > 1$ and the inequalities squeeze the thermodynamical potential to the value:

$$\Omega \approx -\frac{\mu}{6}\left(\frac{\mu}{\hbar\omega_0}\right)^2\left[1 + \pi^2\left(\frac{k_B T}{\mu}\right)^2 - 6\left(\frac{k_B T}{\mu}\right)^3 F_3\big(-\mathrm{e}^{-\frac{\mu}{k_B T}}\big)\right]. \tag{63}$$

For extremely high temperature, $k_B T \gg \mu$, we have $\kappa_\pm \approx 1$ so that the thermodynamic potential is approximately equal to

$$\Omega \approx k_B T\left(\frac{k_B T}{\hbar\omega_0}\right)^2 F_3(-1). \tag{64}$$

This is in agreement with the exact results presented in [5].

4.2. $su(2)$ symmetry

This dynamical symmetry can be put into evidence by introducing the operators $L_+ = a_1^\dagger a_2$ and $L_- = a_2^\dagger a_1$. The commutation relations read

$$[L_+, L_-] = 2\frac{L_z}{\hbar\omega_c}, \quad \left[\frac{L_z}{\hbar\omega_c}, L_\pm\right] = \pm L_\pm \tag{65}$$

and the invariant Casimir operator is given by

$$\mathcal{C} = \frac{1}{2}(L_+L_- + L_-L_+) + \left(\frac{L_z}{\hbar\omega_c}\right)^2 = \left(\frac{N_1+N_2}{2}\right)\left(\frac{N_1+N_2}{2}+1\right). \tag{66}$$

Therefore, for a fixed value $j = (n_1+n_2)/2$ of the operator $(N_1+N_2)/2 = H_0/\hbar\omega - 1/2$, there exists a $(2j+1)$-dimensional UIR of $su(2)$ in which the operator $L_z/(\hbar\omega_c)$ has its spectral values in the range $-j \leqslant m = (n_1-n_2)/2 \leqslant j$. Note that in the weak field limit $\omega_c \ll \omega_0$ the energy levels $E_{n_1,n_2} = \frac{\hbar\omega}{2}(n_1+n_2+1) + \frac{\hbar\omega_c}{2}(n_1-n_2)$ can be approximated by $E_j = \hbar\omega_0(2j+1)$.

This symmetry suggests the use of states (32) that explicitly read

$$\begin{aligned}|J,\gamma,\theta,\varphi\rangle &= \mathrm{e}^{-J/2}\sum_{j\in\mathbb{N}/2}\frac{J^j}{\sqrt{(2j)!}}\,\mathrm{e}^{-\mathrm{i}2j\gamma} \\ &\quad\times\sum_{m=-j}^{j}\sqrt{\binom{2j}{j+m}}\left(\cos\frac{\theta}{2}\right)^{j+m}\left(\sin\frac{\theta}{2}\right)^{j-m}\mathrm{e}^{-\mathrm{i}(j+m)\varphi}|j,m\rangle.\end{aligned} \tag{67}$$

The relations

$$H_0 - \frac{\hbar\omega}{2} = \frac{\hbar\omega}{8\pi^2}\int_0^{2\pi} \mathrm{d}\gamma \int \mathrm{d}S \int_0^\infty \mathrm{d}J\, J\,\frac{J-2}{2}|J,\gamma,\theta,\varphi\rangle\langle J,\gamma,\theta,\varphi|$$
$$L_z = \frac{\hbar\omega_c}{8\pi^2}\int_0^{2\pi} \mathrm{d}\gamma \int \mathrm{d}S \int_0^\infty \mathrm{d}J\,\frac{J^2}{2}\cos\theta|J,\gamma,\theta,\varphi\rangle\langle J,\gamma,\theta,\varphi|$$
$$\langle J,\gamma,\theta,\varphi|H_0|J,\gamma,\theta,\varphi\rangle = \frac{\hbar\omega}{2}(J+1)$$
$$\langle J,\gamma,\theta,\varphi|L_z|J,\gamma,\theta,\varphi\rangle = \hbar\omega_c\frac{J}{2}\cos\theta$$

($\mathrm{d}S = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi$ is the area element of the S^2 sphere) allow us to write lower and upper symbols for the total Hamiltonian:

$$\check{H} = \frac{J}{2}(\hbar\omega + \hbar\omega_c\cos\theta) + \frac{\hbar\omega}{2} = \hat{H} + \hbar\omega. \tag{68}$$

The Berezin–Lieb inequalities

$$\frac{-1}{8\pi^2\beta}\int_0^{2\pi}\mathrm{d}\gamma\int \mathrm{d}S\int_0^\infty \mathrm{d}J\,J\ln[1+\mathrm{e}^{-\beta(\hat{H}-\mu)}] \leqslant \Omega$$
$$\leqslant \frac{-1}{8\pi^2\beta}\int_0^{2\pi}\mathrm{d}\gamma\int \mathrm{d}S\int \mathrm{d}J\,J\ln[1+\mathrm{e}^{-\beta(\check{H}-\mu)}] \tag{69}$$

in this case involve the integral

$$\int \mathrm{d}S\int_0^\infty \mathrm{d}J\,J\ln\left[1+\kappa_\pm\,\mathrm{e}^{-\frac{\beta}{2}(\hbar\omega+\hbar\omega_c\cos\theta)J}\right] = 2\pi\int_{-1}^1\mathrm{d}y\int_0^\infty \mathrm{d}J\,J\ln\left[1+\kappa_\pm\,\mathrm{e}^{-\frac{\beta}{2}(\hbar\omega+\hbar\omega_c y)J}\right] \tag{70}$$

where again $\kappa_\pm = \mathrm{e}^{\beta(\mu\pm\hbar\omega/2)}$ and with the substitution $y = \cos\theta$. Therefore, since

$$\int_0^\infty x\ln(1+k\,\mathrm{e}^{-cx})\,\mathrm{d}x = \frac{-1}{c^2}\begin{cases} F_3(-k) & \text{for} \quad k\leqslant 1, c>0 \\ F_3(-k^{-1}) - \frac{(\ln k)^3}{6} - \frac{\pi^2\ln k}{6} & \text{for} \quad k>1, c>0 \end{cases} \tag{71}$$

and

$$\int_{-1}^1 \frac{\mathrm{d}y}{(\omega+\omega_c y)^2} = \frac{2}{\left(\omega^2-\omega_c^2\right)} = \frac{1}{2\omega_0^2} \tag{72}$$

we can write (69) again as

$$\phi(\kappa_+) \leqslant \Omega \leqslant \phi(\kappa_-) \tag{73}$$

where $\phi(\kappa)$ is given by (62). The fact that we obtained the same result using both families of generalized coherent states is not due to any peculiarity of the Hamiltonian. In fact, the integrals in (59) and in (70) are related through the change of variables

$$|z_1|^2 + |z_2|^2 = J \qquad |z_1|^2 - |z_2|^2 = Jy \tag{74}$$

and therefore will be the same for any Hamiltonian (and not only for the thermodynamic potential).

4.3. $su(1,1)$ symmetry

The $su(1,1)$ structure underlying this 2D magnetism model can be displayed by introducing the operators:

$$K_+ = a_1^\dagger a_2^\dagger \qquad K_- = a_1 a_2 \qquad K_0 = \frac{1}{2}(N_1+N_2+1) = \frac{H_0}{\hbar\omega}. \tag{75}$$

It is easy to see that these operators satisfy the $su(1,1)$ commutation relations. Hence, the Casimir operator reads:

$$\mathcal{D} = K_0^2 - \frac{1}{2}(K_+K_- + K_-K_+) = \left(\frac{N_1 - N_2}{2} + \frac{1}{2}\right)\left(\frac{N_1 - N_2}{2} - \frac{1}{2}\right) = \left(\frac{L_z}{\hbar\omega_c} + \frac{1}{2}\right)\left(\frac{L_z}{\hbar\omega_c} - \frac{1}{2}\right). \quad (76)$$

When $n_1 \geqslant n_2$, for a fixed value $\eta = (n_1 - n_2 + 1)/2 \geqslant 1/2$ of the operator $(N_1 - N_2 + 1)/2$, there exists a UIR of $su(1,1)$ in the discrete series, in which the operator K_0 has its spectral values in the infinite range $\eta, \eta + 1, \eta + 2, \ldots$. Alternatively, when $n_1 \leqslant n_2$, for a fixed value $\rho = (-n_1 + n_2 + 1)/2 \geqslant 1/2$ of the operator $(-N_1 + N_2 + 1)/2$, there also exists a UIR of $su(1,1)$ in which the spectral values of the operator K_0 run in the infinite range $\rho, \rho + 1, \rho + 2, \ldots$.

This symmetry is well suited to the strong field limit, in which $\omega_c \gg \omega_0$ and the energy levels can be approximated by $E_{n_1,n_2} \approx \hbar\omega_c(n_1 + 1/2)$. Therefore, for a given value of n_1, which corresponds to the Landau level index, we have an infinite degeneracy labelled by n_2. One can reinterpret it in terms of $su(1,1)$ symmetry by noting that, for a given value of $(n_1 - n_2)$, the energy eigenstates are ladder states for some discrete series representation of this algebra.

The $su(1,1)$ symmetry has to be explored in a different way from the previous $su(2)$ symmetry. First of all, we have to decompose our Hilbert space into a direct sum of three subspaces or *sectors*, corresponding to $n_1 > n_2, n_1 = n_2$ and $n_1 < n_2$ respectively: $\mathcal{H} = \mathcal{H}_> \oplus \mathcal{H}_= \oplus \mathcal{H}_<$. Accordingly, the trace of a function g of the Hamiltonian decomposes as $\mathrm{Tr}(g(H)) = \mathrm{Tr}_>(g(H)) + \mathrm{Tr}_=(g(H)) + \mathrm{Tr}_<(g(H))$. We can then apply Berezin–Lieb inequalities for each sector.

In the first sector we use as generalized coherent states the superposition (47):

$$|J_1, J_2, \gamma_1, \gamma_2\rangle = \mathrm{e}^{-J_1/2} \sum_{n=0}^{\infty} \frac{J_1^{n/2}}{\sqrt{n!}} \mathrm{e}^{-\mathrm{i}n\gamma_1} (1 - J_2)^{(n+2)/2} \sum_{m=0}^{\infty} \sqrt{\binom{n+m+1}{m}} J_2^{m/2} \mathrm{e}^{-\mathrm{i}m\gamma_2} |n, m\rangle \quad (77)$$

where $n = n_1 - n_2 - 1, m = n_2$.

To the sector $n_1 = n_2 = n$ we simply associate GKS coherent states

$$|J_3, \gamma_3\rangle = \mathrm{e}^{-J_3/2} \sum_{n=0}^{\infty} \frac{J_3^{n/2}}{\sqrt{n!}} \mathrm{e}^{-\mathrm{i}n\gamma_3} |n\rangle. \quad (78)$$

To the last sector we associate states analogous to (77) but with $n = n_2 - n_1 - 1, m = n_1$.

Let us focus on the first sector. The resolution of the projector

$$\int \mathrm{d}\mu(J_1, J_2, \gamma_1, \gamma_2) |J_1, J_2, \gamma_1, \gamma_2\rangle\langle J_1, J_2, \gamma_1, \gamma_2| = \sum_{n_2=0}^{\infty} \sum_{n_1 > n_2} |n_1, n_2\rangle\langle n_1, n_2| \equiv \mathbb{I}_> \quad (79)$$

holds with $\int \mathrm{d}\mu(J_1, J_2, \gamma_1, \gamma_2)$ given by (49) and (50). The restrictions $H_{0>} = \mathbb{I}_> H_0 \mathbb{I}_>$ and $L_{z>} = \mathbb{I}_> L_z \mathbb{I}_>$ of the operators H_0 and L_z can be written as

$$H_{0>} = \hbar\omega \int \mathrm{d}\mu(J_1, J_2, \gamma_1, \gamma_2) \frac{(J_1 - 2)}{2} \left(\frac{1 + J_2}{1 - J_2}\right) |J_1, J_2, \gamma_1, \gamma_2\rangle\langle J_1, J_2, \gamma_1, \gamma_2| \quad (80)$$

$$L_{z>} = \hbar\omega_c \int \mathrm{d}\mu(J_1, J_2, \gamma_1, \gamma_2) \frac{(J_1 - 1)}{2} |J_1, J_2, \gamma_1, \gamma_2\rangle\langle J_1, J_2, \gamma_1, \gamma_2| \quad (81)$$

and their lower symbols are given by

$$\check{H}_{0>} = \hbar\omega\frac{(J_1+2)}{2}\left(\frac{1+J_2}{1-J_2}\right) \qquad \check{L}_{z>} = \hbar\omega_c\frac{(J_1+1)}{2}.$$

Therefore the upper and lower symbols for the restriction of the total Hamiltonian are

$$2\hat{H}_> = \hbar\omega(J_1-2)\left(\frac{1+J_2}{1-J_2}\right) + \hbar\omega_c(J_1-1)$$
$$2\check{H}_> = \hbar\omega(J_1+2)\left(\frac{1+J_2}{1-J_2}\right) + \hbar\omega_c(J_1+1).$$

The lower bound integral in Berezin–Lieb inequalities restricted to the considered sector is then given by

$$-\frac{1}{\beta}\int \mathrm{d}\mu(J_1,J_2,\gamma_1,\gamma_2)\ln[1+\mathrm{e}^{-\beta(\hat{H}_>-\mu)}]$$
$$= -\frac{1}{\beta}\int_0^\infty \mathrm{d}J\int_1^\infty \mathrm{d}y\frac{J}{2}\ln\left[1+\sigma_+(y)\,\mathrm{e}^{-\frac{\beta}{2}(\hbar\omega y+\hbar\omega_c)J}\right] \tag{82}$$

(with the substitution $y=\frac{1+J_2}{1-J_2}$), where

$$\sigma_\pm(y) = \mathrm{e}^{\pm\beta(\hbar\omega y+\hbar\omega_c/2\pm\mu)}. \tag{83}$$

Since $\sigma_+(y)$ is always larger than 1, we get for (82)

$$\frac{2}{\beta^3\hbar^2}\int_1^\infty \frac{\mathrm{d}y}{(\omega y+\omega_c)^2}\left\{F_3(-\sigma_+(y)^{-1}) - \frac{(\ln\sigma_+(y))^3}{6} - \frac{\pi^2\ln\sigma_+(y)}{6}\right\}. \tag{84}$$

This integral is divergent and therefore yields no lower bound to the thermodynamic potential. For the right-hand integral of the Berezin–Lieb inequalities we have

$$-\frac{1}{\beta}\int_0^\infty \mathrm{d}J\int_1^\infty \mathrm{d}y\frac{J}{2}\ln\left[1+\sigma_-(y)\,\mathrm{e}^{-\frac{\beta}{2}(\hbar\omega y+\hbar\omega_c)J}\right]. \tag{85}$$

When $\sigma_-(y)$ is larger than 1 the integral diverges. Therefore we assume $\hbar\omega+\hbar\omega_c/2>\mu$ and then (85) becomes

$$-\frac{2}{\beta^3\hbar^2}\int_1^\infty \mathrm{d}y\frac{F_3(-\sigma_-(y))}{(\omega y+\omega_c)^2} = -\frac{2}{\beta^3\hbar^2}\sum_{n=1}^\infty \frac{(-1)^n\,\mathrm{e}^{-n\beta(\hbar\omega_c/2-\mu)}}{n^3\omega(\omega+\omega_c)}E_2(n\beta\hbar(\omega+\omega_c)) \equiv U \tag{86}$$

where

$$E_n(x) = \int_1^\infty \frac{\mathrm{e}^{-xt}}{t^n}\,\mathrm{d}t. \tag{87}$$

For the sector defined by $n_1=n_2=n$ we have

$$\sum_{n=0}^\infty |n\rangle\langle n| = \frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\gamma\int_0^\infty \mathrm{d}J|J,\gamma\rangle\langle J,\gamma|$$
$$\check{H}_{0=} = \frac{\hbar\omega}{2}(2J+1) = \hat{H}_{0=} + \hbar\omega.$$

The bounds over the whole Hilbert space are obtained by adding the results for each sector. We finally obtain the inequality:

$$\Omega < \psi(\mathrm{e}^{-\beta(\hbar\omega/2-\mu)}) + U \tag{88}$$

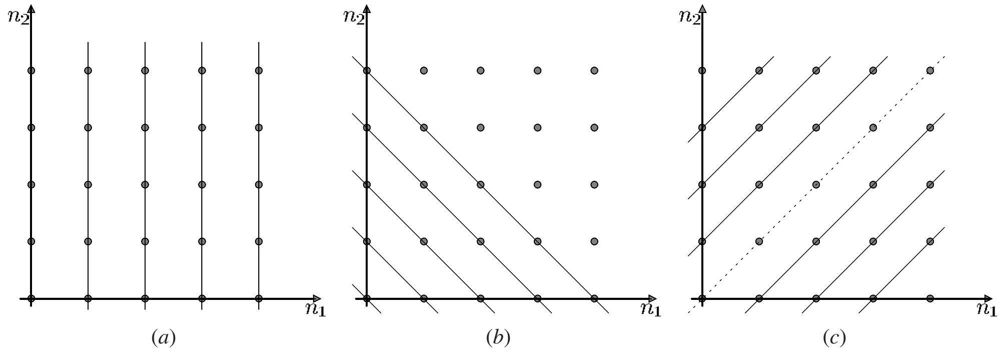

Figure 1. Pictorial representation of the state space $|n_1, n_2\rangle$. (*a*) Representations of 1D harmonic oscillator. (*b*) Irreducible representations of $su(2)$. (*c*) The discrete series of $su(1, 1)$; lines above (below) the dashed line belong to the first (second) sector, as considered in the text.

where $\psi(\kappa)$ is given by

$$\begin{aligned}\psi(\kappa) &= -\frac{1}{2\pi\beta}\int_0^{2\pi} \mathrm{d}\gamma \int_0^{\infty} \ln[1 + \kappa\, \mathrm{e}^{-\beta\hbar\omega J}]\, \mathrm{d}J \\ &= -\frac{1}{\beta^2\hbar\omega}\begin{cases} -F_2(-\kappa) & \text{for} \quad \kappa < 1 \\ F_2(-\kappa^{-1}) + \frac{1}{2}(\ln\kappa)^2 - 2F_2(-1) & \text{otherwise.}\end{cases}\end{aligned}$$

The inequality (88) is quite different from (61), showing that in general the results obtained using different families of coherent states will not be the same. In particular, no lower bound can be established using the $su(1, 1)$ coherent states due to the divergence of the integral (84).

The three different types of coherent states that have been used for this two-dimensional model have a simple geometric interpretation, shown in figure 1. In each one of them we present a different perspective of the Hilbert space of states of the system (to each dot corresponds a state $|n_1, n_2\rangle$), and the involved representations of $su(2)$ and $su(1, 1)$.

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